Exact fermion zero-mode for the new calorons

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We construct the fermion zero-mode for arbitrary charge one $SU(n)$ calorons with non-trivial holonomy, both in the finite temperature context (anti-periodic boundary conditions in time) and in the Kaluza-Klein compactification context (periodic boundary conditions in time). The zero-mode is localised on one of the constituent monopoles and we discuss a relation to the Callias index theorem.

1. Introduction

The $SU(n)$ instantons at finite temperature (or calorons) can be seen as bound states of n constituent monopoles, evident only when the Polyakov loop at spatial infinity is non-trivial. In the periodic gauge, $A_\mu(t+\beta,\vec{x})=A_\mu(t,\vec{x})$,

$$\mathcal{P}_\infty = \lim_{|\vec{x}|\to\infty} P\,\exp(\int_0^\beta A_0(\vec{x},t)dt). \tag{1}$$

After a suitable constant gauge transformation, it can be characterised by $\sum_{m=1}^n \mu_m=0$ and

$$\begin{aligned}&\mathcal{P}_\infty^0 = \exp[2\pi i\,\mathrm{diag}(\mu_1,\ldots,\mu_n)],\\ &\mu_1\leq\ldots\leq\mu_n\leq\mu_{n+1}\equiv 1+\mu_1.\end{aligned} \tag{2}$$

Using the classical scale invariance we can always arrange $\beta=1$, as will be assumed throughout. A remarkably simple formula for the $SU(n)$ action density exists [1,2].

$$\begin{aligned}&\mathrm{Tr}F_{\mu\nu}^2(x) = \partial_\mu^2\partial_\nu^2\log\psi(x),\\ &\psi(x) = \tfrac{1}{2}\mathrm{tr}(\mathcal{A}_n\cdots\mathcal{A}_1)-\cos(2\pi t),\\ &\mathcal{A}_m \equiv \frac{1}{r_m}\begin{pmatrix} r_m & |\vec{y}_m-\vec{y}_{m+1}| \\ 0 & r_{m+1}\end{pmatrix}\begin{pmatrix} c_m & s_m \\ s_m & c_m\end{pmatrix},\end{aligned} \tag{3}$$

with $r_m=|\vec{x}-\vec{y}_m|$ the center of mass radius of the m^{th} constituent monopole, which can be assigned a mass $8\pi^2\nu_m$, where $\nu_m\equiv\mu_{m+1}-\mu_m$. Furthermore, $c_m\equiv\cosh(2\pi\nu_m r_m)$, $s_m\equiv\sinh(2\pi\nu_m r_m)$, $r_{n+1}\equiv r_1$ and $\vec{y}_{n+1}\equiv\vec{y}_1$.

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2. Monopole constituents

These generalised caloron solutions can be found [3] using a combination of the Nahm transformation [4] and the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction [5]. The latter is mainly needed to resolve the delta function singularities that arise in the Nahm transformation, although other methods were developed as well [6].

The Nahm equation for these charge one instantons reduces to an abelian problem on the circle, parametrised by $z \bmod 1$,

$$\frac{d}{dz}\hat{A}_j(z) = 2\pi i\sum_m \left(y_m^j - y_{m-1}^j\right)\delta(z-\mu_m), \tag{4}$$

giving $\hat{A}_j(z)=2\pi i y_m^j$, for $z\in[\mu_m,\mu_{m+1}]$. In the monopole literature $\hat{A}_j(z)$ is usually denoted by $T_j(z)$. Taking one interval in isolation, applying the Nahm transformation [4] gives a single static Bogomol'nyi-Prasad-Sommerfeld (BPS) monopole with mass proportional to the length (ν_m) of the interval. Taking $|\vec{y}_n|\to\infty$ leaves the interval $[\mu_1,\mu_n]$, allowing for the interpretation of an $SU(n)$ monopole with μ_i specifying the eigenvalues of the Higgs field at infinity, for which it is crucial they add to zero. Indeed, in the periodic gauge A_0 tends to a constant at spatial infinity.

Note that we have to order $\exp(2\pi i\mu_m)$ along the circle to ensure that the ν_i add to 1, an ordering inherited by μ_m when extended to the real line by insisting $\mu_{kn+m}=k+\mu_m$, for any

integer k. Let us pick one to be labelled by μ_1. All we can guarantee at this point is that $\sum_{m=1}^n \mu_m = \ell$, an integer. With $\mu_{kn+m} = k + \mu_m$, we find $\sum_{m=1}^n \mu_{m+p} = \ell + p$, for *any* integer p. A cyclic shift of the labels by $p = -\ell$ proves that there is a *unique* choice of the μ_m that satisfy eq. (2). It demonstrates why $\vec{y}_n$ does play a special role, and in the limit $|\vec{y}_n| \to \infty$ one therefore has a *static* monopole solution [4], which can be seen as the composite of $n-1$ BPS monopoles of mass ν_m, located at $\vec{y}_m$, for $m = 1, \cdots, n-1$. From the general formalism it is clear these $n-1$ monopole constituents are time independent, as was verified explicitly for $SU(2)$ [2,3]. Note that our argument demonstrates that for $|\vec{y}_m| \to \infty$ with $m \neq n$, one is left with a gauge field that cannot be time independent, even though the resulting action density is [2].

The significance of one constituent carrying a time dependent field lies in the fact that the n constituent monopoles form an instanton, and the topological charge can be associated to the so-called Taubes-winding [7], described by a time dependent (gauge) rotation, going full circle when t progresses over one period. For $SU(2)$ this can be read-off from the explicit expression for the gauge field [2,3]. We thus conclude that the constituent located at $\vec{y}_n$ is the one that carries this Taubes-winding, even though its action density is time independent for well-separated constituents. This conclusion can also be drawn from the formalism developed in ref. [6], see also ref. [8].

3. Fermion zero-mode

The basic ingredient in the construction of caloron solutions is a Greens function defined by

$$\left(D_z^2 + r^2(x;z) + \sum_m \delta_m(z)\right)\hat{f}_x(z,z') = \delta(z-z'), \quad (5)$$

where $D_z = (2\pi i)^{-1}\partial_z - t$, $r^2(x;z) = r_m^2(x)$ for $z \in [\mu_m, \mu_{m+1}]$ and $\delta_m(z) = \delta(z-\mu_m)|\vec{y}_m - \vec{y}_{m-1}|/2\pi$. A similarity with the impurity scattering problem allows for a straightforward solution [1], which we present here for the case that $\mu_m \leq z' \leq z \leq \mu_{m+1}$ (extended to $z < z'$ by $\hat{f}_x(z',z) = \hat{f}_x^*(z,z')$)

$$\hat{f}_z(z,z') = \frac{\pi e^{2\pi i t(z-z')}}{r_m \psi}\Big(e^{-2\pi i t}\sinh\left(2\pi(z-z')r_m\right) + \\ <v_m(z')|\mathcal{A}_{m-1}\cdots\mathcal{A}_1\mathcal{A}_n\cdots\mathcal{A}_m|w_m(z)>\Big), \quad (6)$$

where the spinors v_m and w_m are defined by

$$\begin{aligned} v_m^1(z) &= -w_m^2(z) = \sinh\left(2\pi(z-\mu_m)r_m\right), \\ v_m^2(z) &= \quad w_m^1(z) = \cosh\left(2\pi(z-\mu_m)r_m\right). \end{aligned} \quad (7)$$

For the zero-mode densities we find

$$|\Psi_z(x)|^2 = -(2\pi)^{-2}\partial_\mu^2 \hat{f}_x(z,z), \quad (8)$$

derived exactly as for $SU(2)$ [9], not repeated here. With the gauge field in the periodic gauge one has, $\Psi_z(t+1,\vec{x}) = \exp(2\pi i z)\Psi_z(t,\vec{x})$. To obtain the finite temperature fermion zero-mode one puts $z = \frac{1}{2}$, whereas for the fermion zero-mode with periodic boundary conditions, relevant in supersymmetric applications, one takes $z = 0$.

In figure 1 we show a typical $SU(3)$ caloron, illustrating that also for $n > 2$ the fermion zero-modes are localised on one of the constituents. This localisation can be established easily in the

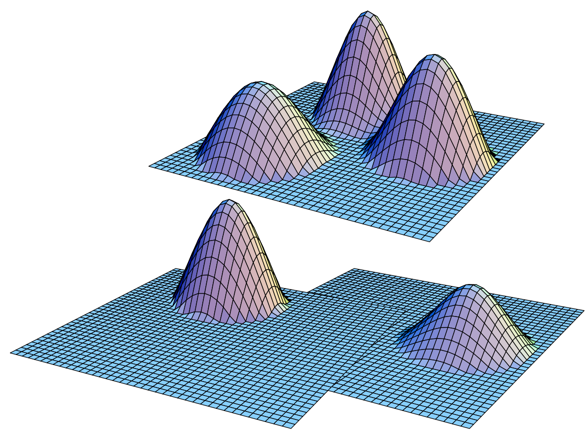

Figure 1. The action densities (top) for the $SU(3)$ caloron, cut off at $1/(2e)$, on a logarithmic scale, with $(\mu_1,\mu_2,\mu_3) = (-17,-2,19)/60$ for $t=0$ in the plane defined by $\vec{y}_1 = (-2,-2,0)$, $\vec{y}_2 = (0,2,0)$ and $\vec{y}_3 = (2,-1,0)$, for $\beta = 1$, with masses $8\pi^2\nu_i$, $(\nu_1,\nu_2,\nu_3) = (0.25, 0.35, 0.4)$. On the bottom-left is shown the zero-mode density for fermions with anti-periodic boundary conditions in time and on the bottom-right for periodic boundary conditions, at equal logarithmic scales, cut off below $1/e^5$.

limit of large $|\vec{y}_i - \vec{y}_{i+1}|$ for all i, in which case one finds, when $z \in [\mu_m, \mu_{m+1}]$,

$$\hat{f}_x(z,z) = \frac{\sinh[2\pi(z-\mu_m)r_m]\sinh[2\pi(\mu_{m+1}-z)r_m]}{r_m \sinh[2\pi\nu_m r_m]/2\pi}$$

making explicit that the location of the zero-mode is determined by the interval that contains the appropriate value of z. From eq. (2) it follows that $\mu_1 \leq 0 \leq \mu_n$, such that the periodic zero-mode is associated to the *static* constituent at $\vec{y}_m$, with $\mu_m \leq 0 \leq \mu_{m+1}$. This is precisely the condition for the existence of a zero-mode given by the Callias index theorem [10] (see also the appendix of ref. [11]). Due to the static background (for well-separated constituents), time dependence of the zero-mode would be of the form $\exp(2\pi ikt)$ for k integer, shifting $z=0$ by k, out of the interval that allows for a zero-mode.

Allowing for $k = \pm\frac{1}{2}$, for which $\exp(2\pi ikt)$ turns the periodic zero-mode anti-periodic, we can have situations where this anti-periodic zero-mode is associated to one of the static monopole constituents. A specific example for $SU(3)$ where this occurs is $(\mu_1,\mu_2,\mu_3) = (-0.48,-0.03,0.51)$, yielding $(\nu_1,\nu_2,\nu_3) = (0.45,0.54,0.01)$. *Both* the periodic and the anti-periodic zero-mode are associated to the 2$^{\rm nd}$ constituent. We note that, apart from the fact that the 3$^{\rm rd}$ constituent is nearly massless, both zero-modes are very broad since $\min(z-\mu_2,\mu_3-z) = 0.03$ for $z=0$ and 0.01 for $z=\frac{1}{2}$. For $SU(2)$ $z=0$ is always midway between μ_1 and μ_2 and $z=\frac{1}{2}$ midway between μ_2 and $\mu_3 = 1+\mu_1$). When z coincides with μ_i, the zero-mode is no longer normalisable, which is the origin of the delta function singularities in the Nahm transformation.

4. Conclusions

In conclusion, for well-separated constituents the fermion zero-mode is localised to a single constituent. For $SU(2)$ the anti-periodic zero-mode is always associated to the constituent that carries Taubes-winding [9]. For $SU(n>2)$ this is also typically true (see fig. 1), in particular when well localised, something that may be significant for developing a model for the QCD vacuum that combines monopoles and instantons [2,3,9]. However, exceptions exist where both the periodic and anti-periodic zero-mode are associated to (possibly the same) static constituent(s), although this tends to be accompanied by nearly massless constituents, and rather delocalised zero-modes.

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